

CHARACTERIZATIONS OF PROJECTIVE SPACES AND HYPERQUADRICS VIA POSITIVITY PROPERTIES OF THE TANGENT BUNDLE

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ABSTRACT. Let X be a smooth complex projective variety. A recent conjecture of S. Kovács states that if the p^{th} -exterior power of the tangent bundle T_X contains the p^{th} -exterior power of an ample vector bundle, then X is either a projective space or a smooth quadric hypersurface. This conjecture is appealing since it is a common generalization of Mori's, Wahl's, Andreatta-Wisniewski's, Kobayashi-Ochiai's and Araujo-Druel-Kovács's characterizations of these spaces. In this paper I give a proof affirming this conjecture for varieties with Picard number 1.

1. INTRODUCTION

Let X be a smooth complex projective variety of dimension n . In a seminal paper [Mor79], S. Mori proved that the only such varieties having ample tangent bundle T_X are projective spaces. This result finally settled Hartshorne's conjecture [Har70], the algebraic analog of Frankel's conjecture [Fra61] in complex differential geometry. (Another proof of Frankel's conjecture was given around the same time by Y. Siu and S. Yau in [SY80] using harmonic maps.) Since then, the ideas of [Mor79] have been expanded significantly, and there are many results in the literature using positivity properties of T_X to characterize projective spaces and quadric hypersurfaces. In this paper I will prove another characterization in this direction:

Theorem 1.1. *Let X be a smooth complex projective variety of dimension n with Picard number 1. Assume that there exists an ample vector bundle \mathcal{E} of rank r on X and a positive integer $p \leq r$ such that $\wedge^p \mathcal{E} \subseteq \wedge^p T_X$. Then either $X \simeq \mathbb{P}^n$, or $p = n$ and $X \simeq Q_p \subset \mathbb{P}^{p+1}$, where Q_p denotes a smooth quadric hypersurface in \mathbb{P}^{p+1} .*

Theorem 1.1 gives an affirmative answer for varieties with Picard number 1 of the following more general conjecture of S. Kovács:

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Conjecture 1.2 (Kovács). *Let X be a smooth complex projective variety of dimension n . If there exists an ample vector bundle \mathcal{E} of rank r on X and a positive integer $p \leq r$ such that $\wedge^p \mathcal{E} \subseteq \wedge^p T_X$, then either $X \simeq \mathbb{P}^n$, $p = n$ and $X \simeq Q_p \subset \mathbb{P}^{p+1}$, where Q_p denotes a smooth quadric hypersurface in \mathbb{P}^{p+1} .*

Motivation for this conjecture comes from the desire to unify existing characterization results of this type into a single statement. Mori's proof of the Hartshorne conjecture in 1979 was the first major result, and its method of studying rational curves of minimal degree has been a catalyst for much that has followed.

Theorem 1.3. [Mor79] *Let X be a smooth complex projective variety of dimension n , and assume that the tangent sheaf T_X is ample. Then $X \simeq \mathbb{P}^n$.*

In 1983, J. Wahl proved a related statement using algebraic methods:

Theorem 1.4. [Wah83] *Let X be a smooth complex projective variety of dimension n , and assume that the tangent sheaf T_X contains an ample line bundle \mathcal{L} . Then either $(X, \mathcal{L}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ or $(X, \mathcal{L}) \simeq (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$.*

Note that S. Druel gave a geometric proof of this theorem in [Dru04]. In 1998, F. Campana and T. Peternell generalized Wahl's theorem to bundles of rank $r = n, n - 1$, and $n - 2$ [CP98]. Finally, in 2001, M. Andreatta and J. Wiśniewski proved the analogous statement for vector bundles of arbitrary rank:

Theorem 1.5. [AW01] *Let X be a smooth complex projective variety of dimension n , and assume that the tangent sheaf T_X contains an ample vector bundle \mathcal{E} of rank r . Then either $(X, \mathcal{E}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r})$ or $r = n$ and $(X, \mathcal{E}) \simeq (\mathbb{P}^n, T_{\mathbb{P}^n})$.*

It is worth noting that in 2006 C. Araujo developed a different approach to Theorem 1.5 using the variety of minimal rational tangents [Ara06]. In 1973, S. Kobayashi and T. Ochiai proved the following theorem characterizing both projective spaces and quadric hypersurfaces:

Theorem 1.6. [KO73] *Let X be an n -dimensional compact complex manifold with ample line bundle \mathcal{L} . If $c_1(X) \geq (n + 1)c_1(\mathcal{L})$ then $X \simeq \mathbb{P}^n$. If $c_1(X) = nc_1(\mathcal{L})$ then $X \simeq Q_n$, where $Q_n \subseteq \mathbb{P}^{n+1}$ is a hyperquadric.*

Most recently, the following conjecture of A. Beauville [Bea00] was verified by Araujo, Druel, and Kovács:

Theorem 1.7. [ADK08] *Let X be a smooth complex projective variety of dimension n , and let \mathcal{L} be an ample line bundle on X . If $H^0(X, \wedge^p T_X \otimes \mathcal{L}^{-p}) \neq 0$ for some positive integer p , then either $(X, \mathcal{L}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ or $p = n$ and $(X, \mathcal{L}) \simeq (Q_p, \mathcal{O}_{Q_p}(1))$, where Q_p denotes a smooth quadric hypersurface in \mathbb{P}^{p+1} .*

Theorems 1.3–1.7 are comparable in their direction but incongruous in the sense that no one of them implies all the others. Conjecture 1.2 is appealing since it simultaneously implies all of them: Mori’s theorem is covered by the case $p = 1$, $\mathcal{E} = T_X$, Wahl’s theorem by $p = 1$, $r = 1$, and the result of Andreatta-Wisniewski by taking $p = 1$. The main theorem of [ADK08] is covered by setting $\mathcal{E} = \mathcal{L}^{\oplus r}$ where $r = p$, and [KO73] by setting $\mathcal{E} = \mathcal{L}^{\oplus n}$ and $\mathcal{E} = \mathcal{L}^{\oplus n-1} \oplus \mathcal{L}^{\otimes 2}$.

Remark 1.8. Notice that 1.2 also generalizes 1.7 to the case where $\wedge^p T_X$ contains a product of p distinct ample line bundles: $(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_p) \subseteq \wedge^p T_X$.

It is easy to check that Conjecture 1.2 holds in some simple cases, for example, when the dimension of X is small: If $\dim X = 1$, the only choice for the integer p is $p = 1$. In this case, Conjecture 1.2 follows from Theorem 1.4 (and also Theorem 1.5.) When $\dim X = 2$, Conjecture 1.2 follows easily from the following theorem:

Theorem 1.9. *Let X be a smooth complex projective variety of dimension 2, and assume that $-K_X = A + F$ where F is an effective divisor and A is an ample divisor such that $A \cdot C \geq 2$ for every smooth rational curve $C \subseteq X$, $C \simeq \mathbb{P}^1$. Then either $X \simeq \mathbb{P}^2$ or $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. First notice that X has negative Kodaira dimension since $-K_X \cdot C > 0$ for every general curve $C \subseteq X$. Let $X \rightarrow X_{\min}$ be a minimal model obtained by blowing down sufficiently many (-1) -curves. Since $\kappa(X) < 0$, X_{\min} is isomorphic to either \mathbb{P}^2 or a ruled surface over a curve B . Before addressing each case, I prove the following claim that will be used in the rest of the proof:

Claim 1.9.1. *Let X , F , and A be as in the statement of Theorem 1.9 above. If $C \subseteq X$ is a curve such that $C \simeq \mathbb{P}^1$ and $C^2 < 0$, then $F \cdot C < 0$ and hence $C \subseteq F$.*

Proof. The following computation implies the claim:

$$F \cdot C = (-K_X - A) \cdot C = (-K_X \cdot C) - (A \cdot C) \leq (2 + C^2) - 2 < 0$$

Here the first inequality follows from adjunction and the initial assumption on the ample divisor A . \square

Continuing with the proof of Theorem 1.9, assume that $X \not\simeq \mathbb{P}^2$. It follows that X admits a morphism to a ruled surface $Y \rightarrow B$: If $X_{\min} \simeq \mathbb{P}^2$ then Y is the blow-up of \mathbb{P}^2 at a single point. Otherwise $Y \simeq X_{\min}$. The ruling $Y \rightarrow B$ induces a morphism $\pi : X \rightarrow B$. I will show by contradiction that the fibers of π are irreducible, hence X itself is ruled: Suppose that G is a reducible fiber of π . Then G may be written as a sum $G = \sum G_i$ where $G_i \simeq \mathbb{P}^1$ and $G_i^2 < 0$. By 1.9.1, each G_i (and hence G) is contained in the effective divisor F . Also, as G is a fiber, $G \cdot G_i = 0$. It follows from 1.9.1 that $(F - G) \cdot G_i < 0$ for each G_i , therefore G must be contained in $F - G$, i.e., F contains $2G$. Repeating this computation, one may show that $nG \subseteq F$ for any positive integer n , but this is a contradiction since F is a fixed effective divisor. Therefore the fibers of π are irreducible as claimed, and $\pi : X \rightarrow B$ itself must be a ruling of X .

Using the notation of [Har77, V.2.8], there exists a distinguished locally free sheaf \mathcal{E}' of rank 2 and degree $-e$ such that $X \simeq \mathbb{P}(\mathcal{E}')$. Furthermore, in this case there is a section $\sigma : B \rightarrow X$ with image C_0 such that $\mathcal{L}(C_0) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E}')} (1)$. Continuing with the notation of [Har77, V.2], let \mathfrak{f} be a fiber of π . In particular, recall that $C_0 \cdot \mathfrak{f} = 1$ and $\mathfrak{f}^2 = 0$. By the assumption on A and the fact that \mathfrak{f} is nef, one has: $-K_X \cdot \mathfrak{f} = A \cdot \mathfrak{f} + F \cdot \mathfrak{f} \geq 2$. On the other hand, by [Har77, V.2.11], $-K_X \cdot \mathfrak{f} = 2$. Therefore $A \cdot \mathfrak{f} = 2$ and $F \cdot \mathfrak{f} = 0$, and the latter inequality implies that $F = m\mathfrak{f}$ is nef. It follows that $-K_X$ is ample, (it is the sum of an ample and a nef divisor), and therefore X is a Del Pezzo surface. This means that X is both ruled and rational, hence it is a Hirzebruch surface, i.e., \mathcal{E}' is decomposable. By [Har77, 2.12], it follows that $e \geq 0$. On the other hand, since $C_0 \not\subseteq F$, 1.9.1 implies that $C_0^2 \geq 0$. But $e = -C_0^2$ by [Har77, V.2.9], therefore $e = C_0^2 = 0$. The only Hirzebruch surface with $e = 0$ is $\mathbb{P}^1 \times \mathbb{P}^1$, and this completes the proof of Theorem 1.9. \square

Corollary 1.10. *Conjecture 1.2 holds when $\dim X = 2$.*

Proof. If $\dim X = 2$, there are two choices for the integer p . If $p = 1$, Conjecture 1.2 follows from Theorem 1.4, so we may assume that $p = 2$. Over a field of characteristic zero, the wedge product of an ample vector bundle is again ample [Har66, 5.3], so the condition $\wedge^2 \mathcal{E} \subseteq \wedge^2 T_X$ implies that ω_X^{-1} contains an ample line bundle. In particular, one may write $-K_X = A + F$ where $A = c_1(\wedge^2 \mathcal{E})$ is the corresponding ample divisor and F is an effective divisor. Notice that $A \cdot C \geq 2$ for every smooth rational curve $C \subseteq X$, $C \simeq \mathbb{P}^1$: Since \mathcal{E} is ample, the degree of $\mathcal{E}|_C = A|_C$ is bounded below by the rank of \mathcal{E} . Now Theorem 1.7 shows that Conjecture 1.2 holds when $\dim X = 2$. \square

In this paper I will show that Conjecture 1.2 holds for all varieties with Picard number 1. The paper is organized as follows: Section 2 is devoted to gathering necessary definitions and results about minimal covering families of rational curves. Section 3 will cover some auxiliary results needed for the main proof. The proof of Theorem 1.1 is covered in Section 4.

Notation: I will follow the notation of [Kol96] in the discussion of rational curves. By a vector bundle I mean a locally free sheaf; a line bundle is an invertible sheaf. I will denote by $\mathbb{P}(V)$ the natural projectivization of a vector space V . A point $x \in X$ is general if it is contained in a dense open subset of $U \subseteq X$ where U is a fixed open subset determined by the context. Throughout the paper I will be working over the field of complex numbers.

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Note: Upon completion of this paper, I learned of a somewhat related result by Matthieu Paris [Par10].

2. RATIONAL CURVES OF MINIMAL DEGREE ON UNIRULED VARIETIES

The proof of the main theorem relies on studying rational curves of minimal degree on X . Starting with [Mor79], many tools have been developed for analyzing families of rational curves on uniruled varieties; for the reader's convenience I summarize the most important developments here.

Let X be a smooth complex projective variety. If X is uniruled, one can find an irreducible component $H \subset \text{RatCurves}^n(X)$ such that the natural map $\text{Univ}_H \rightarrow X$ is dominant. Such a component is called a *dominating family* of rational curves on X . The component H is called *unsplit* if it is proper, and is called *minimal* if the subfamily of curves parameterized by H passing through a general point $x \in X$ is proper. A uniruled variety always admits a minimal dominating family of curves [Kol96, IV.2.4].

If $C \subset X$ is a rational curve on X and $f : \mathbb{P}^1 \rightarrow C \subseteq X$ is its normalization, the corresponding point in $\text{RatCurves}^n(X)$ is denoted by $[f]$. If H is a minimal dominating family, then the splitting type of

f^*T_X for any general $[f] \in H$ is:

$$f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$$

where $d := \deg(f^*T_X) - 2 \geq 0$ [Kol96, IV.2.9, IV.2.10]. The “positive part” of f^*T_X is the subbundle defined by:

$$(f^*T_X)^+ := \text{im}[H^0(\mathbb{P}^1, f^*T_X(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow f^*T_X] \hookrightarrow f^*T_X.$$

If H is a fixed minimal dominating family of rational curves on X , one can define an equivalence relation on the points of X via H : Two points $x_1, x_2 \in X$ are *H-equivalent* if they can be connected by a chain of rational curves parameterized by H . By [Kol96, IV.4.16], there exists a proper surjective morphism $\pi^\circ : X^\circ \rightarrow Y^\circ$ from a dense open subset $X^\circ \subseteq X$ onto a normal variety Y° whose fibers are *H*-equivalence classes. The morphism π° is often called the *H-rationally connected quotient* of X . If Y° is a point, then X is called *H-rationally connected*. An important fact used later is that when the Picard number of X is 1, the *H*-rationally connected quotient is trivial:

Proposition 2.1. *Let X be a smooth complex projective variety, H a minimal dominating family of rational curves on X , and $\pi^\circ : X^\circ \rightarrow Y^\circ$ the corresponding *H*-rationally connected quotient. If $\rho(X) = 1$, then Y° is a point.*

Proof. Suppose that Y° is positive dimensional. Let D_{Y° be an ample effective divisor on Y° , D_{X° its pullback on X° and D_X the closure of D_{X° in X . Since $\rho(X) = 1$, every effective divisor is ample, and it follows that every rational curve parameterized by H has positive intersection with D_X . Let C be a rational curve parameterized by H and contained in X° . By definition, π° contracts C and hence $D_{X^\circ} \cdot C = 0$, a contradiction. Therefore Y° must be a point. \square

Remark 2.2. The converse of Proposition 2.1 is also true by [Kol96, IV.3.13.3] if one assumes additionally that H is unsplit, but this will not be needed here.

Remark 2.3. The equivalence relation above can be extended to a collection of families of rational curves H_1, H_2, \dots, H_k : Two points $x_1, x_2 \in X$ are (H_1, H_2, \dots, H_k) -equivalent if they can be connected by a chain of rational curves parameterized by H_1, H_2, \dots, H_k . This induces a morphism on a dense open subset of X with (H_1, H_2, \dots, H_k) -rationally connected fibers, called the (H_1, H_2, \dots, H_k) -rationally connected quotient of X .

It is worth noting that a minimal dominating family H may not always restrict to a minimal dominating family on the fibers of the

H -rationally connected quotient. To be precise, if X_y is a fiber of an H -rationally connected quotient of X and ι is the natural map

$$(2.3.1) \quad \iota : \text{RatCurves}^n(X_y) \hookrightarrow \text{RatCurves}^n(X)$$

it is not always the case that $\iota^{-1}(H) \subseteq \text{RatCurves}^n(X_y)$ is irreducible:

Example 2.4. Let $Y \subseteq \mathbb{P}^9$ be the open subset parameterizing smooth quadric surfaces in \mathbb{P}^3 , X the corresponding open subset of the universal hypersurface in $\mathbb{P}^3 \times \mathbb{P}^9$, $\pi_1 : X \rightarrow \mathbb{P}^3$ and $\pi_2 : X \rightarrow Y \subseteq \mathbb{P}^9$ the restrictions of the usual projection morphisms. Let C be a rational curve on X corresponding to a line on a smooth quadric in \mathbb{P}^3 . (In other words, C has the property of being contracted by π_2 and having image equal to a line under π_1 .) Let $H \subseteq \text{RatCurves}^n(X)$ be the irreducible component containing the point parameterizing C .

I claim that H is in fact a dominating family on X : First notice that H parameterizes all the rational curves in X that correspond to a line on a smooth quadric in \mathbb{P}^3 . Indeed, if C' is any other rational curve with these properties, there exists a smooth deformation of C to C' in X : The images of C and C' in \mathbb{P}^3 are lines, say L and L' , and in \mathbb{P}^3 there exists a smooth deformation of L to L' by a family of lines $\{L_t\}$ parameterized by \mathbb{P}^1 . One can extend this to a family of smooth quadrics $\{Q_t\}$ parameterized over the same base such that $L_t \subset Q_t$ for each $t \in \mathbb{P}^1$. (For example, let Q be the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding, and let L be a distinguished line on Q . There exists a one-parameter family of automorphisms $\{\alpha_t\}$ of \mathbb{P}^3 such that $\alpha_t(L) = L_t$ for each $t \in \mathbb{P}^1$, (just choose an appropriate non-trivial morphism $\mathbb{P}^1 \rightarrow \text{Aut}(\mathbb{P}^3)$), and now the family $\{Q_t := \alpha_t(Q) \mid t \in \mathbb{P}^1\}$ has the desired properties.) Since X is covered by the rational curves corresponding to the lines on the smooth quadrics of \mathbb{P}^3 , H is a dominating family on X .

Next notice that the H -rationally connected quotient is just $\pi_2 : X \rightarrow Y$: On one hand, by construction, every rational curve parameterized by H is contained in a fiber of π_2 . On the other hand, the fibers of π_2 are just the smooth quadrics in \mathbb{P}^3 and each is rationally connected by the lines it contains.

Finally, observe that the restriction of H to any fiber cannot be a minimal dominating family: There are two minimal dominating families on any $\mathbb{P}^1 \times \mathbb{P}^1$, (namely the two families of lines), and the restriction of H to any fiber will contain both of them.

Remark 2.5. The above example also shows that one cannot assume in general that the fibers of the H -rationally connected quotient have

Picard number 1, even when H is unsplit. A necessary condition on H for the fibers to have Picard number 1 is given by [ADK08, 2.3].

Next, recall the definition of the *variety of minimal rational tangents*: If $x \in X$ is a general point of X , let H_x denote the normalization of the subscheme of H parameterizing curves passing through $x \in X$. For general $x \in X$, H_x is a smooth projective variety of dimension $d := \deg(f^*T_X) - 2$ [Kol96, II.1.7, II.2.16]. There exists a map $\tau_x : H_x \dashrightarrow \mathbb{P}(T_x X)$ called the *tangent map* defined by sending a curve that is smooth at $x \in X$ to its corresponding tangent direction at x . The closure of the image of τ_x in $\mathbb{P}(T_x X)$ is called the *variety of minimal rational tangents* at x and is denoted $\mathcal{C}_x \subseteq \mathbb{P}(T_x X)$. The tangent map is actually the normalization morphism of \mathcal{C}_x , a fact proved by S. Kebekus [Keb02] and J. Hwang and N. Mok [HM04]:

Theorem 2.6.

(2.6.1) [Keb02] *The tangent map $\tau_x : H_x \dashrightarrow \mathcal{C}_x$ is a finite morphism.*

(2.6.2) [HM04] *The tangent map $\tau_x : H_x \dashrightarrow \mathcal{C}_x$ is birational, hence it is the normalization.*

The variety \mathcal{C}_x has a natural embedding into $\mathbb{P}(T_x X)$, and this embedding yields important geometric information about X . For example, Araujo shows that when \mathcal{C}_x is a linear subspace of $\mathbb{P}(T_x X)$, the H -rationally connected quotient of X is a projective space bundle:

Theorem 2.7. [Ara06, 1.1] *Assume that \mathcal{C}_x is a d -dimensional linear subspace of $\mathbb{P}(T_x X)$ for a general point $x \in X$. Then there is a dense open subset X° of X and \mathbb{P}^{d+1} -bundle $\varphi^\circ : X^\circ \rightarrow T^\circ$ such that any curve from H meeting X° is a line on a fiber of φ° .*

Lastly, note that the tangent space of \mathcal{C}_x at a point $\tau_x([f])$ is related to the splitting type of f^*T_X in an important way. In particular, the tangent space of \mathcal{C}_x at the point $\tau_x([f])$ is cut out by the positive directions of f^*T_X at $x \in X$:

Lemma 2.8. [Hwa01, 2.3] *Let $[f] \in H$ be a general member, and let $T_x X_f^+ \subseteq T_x X$ be the $(d+1)$ -dimensional subspace corresponding to the positive factors of the splitting $f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$. Then $\mathbb{P}(T_x X_f^+)$ is the projectivized tangent space of \mathcal{C}_x at the point $\tau_x([f])$.*

3. PRELIMINARY RESULTS

Before proving the main theorem, I prove a few auxillary results. In particular, I will show that with the assumptions made in the statement

of Theorem 1.1, X admits a nice cover of rational curves, and one can determine the splitting type of the ample vector bundle \mathcal{E} when restricted to these rational curves.

Lemma 3.1. *Let X be a smooth complex projective variety, \mathcal{E} an ample vector bundle of rank r on X , and assume that $\wedge^p \mathcal{E} \subseteq \wedge^p T_X$ for some positive integer $p \leq r$. Then X is uniruled.*

Proof. Uniruledness of X follows almost immediately from a theorem of Miyaoka, that says that if Ω_X is not generically semipositive, then X is uniruled [Miy87, 8.6]. Since generic semipositivity of Ω_X implies generic semipositivity of $\wedge^p \Omega_X$, it is enough to check that $\wedge^p \Omega_X$ is not generically semipositive: Let C be a general complete intersection curve on X . Then $(\wedge^p \mathcal{E})|_C$ has positive degree since $\wedge^p \mathcal{E}$ is ample. The dual of the inclusion $(\wedge^p \mathcal{E})|_C \hookrightarrow (\wedge^p T_X)|_C$ yields the desired result. \square

Now let $H \subset \text{RatCurves}^n(X)$ be a minimal dominating family of rational curves on X guaranteed by Lemma 3.1. The next lemma determines the splitting type of $f^* \mathcal{E}$ for $[f] \in H$.

Lemma 3.2. *Let X be a smooth complex projective variety, \mathcal{E} an ample vector bundle of rank r on X , and $p \leq r$ a positive integer such that $\wedge^p \mathcal{E} \subseteq \wedge^p T_X$. Let H be a minimal dominating family of rational curves on X . Then either $f^* \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}$ for every $[f] \in H$, or $f^* \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$ for every $[f] \in H$.*

Proof. First let $[f] \in H$ be a general member of H . Since \mathcal{E} is ample and $[f]$ parameterizes a rational curve, $f^* \mathcal{E}$ splits as a direct sum of positive degree line bundles:

$$f^* \mathcal{E} \simeq \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(\alpha_i), \quad \alpha_i \geq 1.$$

It follows that $f^*(\wedge^p \mathcal{E})$ splits as a sum of line bundles of degree at least p :

$$f^*(\wedge^p \mathcal{E}) \simeq \bigoplus_{j=1}^{\binom{r}{p}} \mathcal{O}_{\mathbb{P}^1}(\beta_j), \quad \beta_j = \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_p} \geq p.$$

By assumption,

$$\begin{aligned} f^*(\wedge^p \mathcal{E}) &\subseteq f^*(\wedge^p T_X) \simeq \wedge^p (\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}) \\ &\simeq \mathcal{O}_{\mathbb{P}^1}(p+1)^{\oplus q_1} \oplus \mathcal{O}_{\mathbb{P}^1}(p)^{\oplus q_2} \oplus \cdots \end{aligned}$$

and the highest degree line bundle occurring on the right is $\mathcal{O}_{\mathbb{P}^1}(p+1)$. Therefore $p \leq \beta_j \leq p+1$ for each $1 \leq j \leq \binom{r}{p}$, but this leaves

only two possibilities for $f^*\mathcal{E}$: Either $f^*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}$ or $f^*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$.

Lastly, observe that \mathcal{E} must split the same way on every rational curve parameterized by H : Since H is an irreducible component of $\text{RatCurves}^n(X)$, the intersection number of a fixed line bundle on X and any curve C parameterized by H is independent of C . In particular, the degree of $\det(\mathcal{E})$ remains constant on all the rational curves parameterized by H , and it follows that $\deg(f^*\mathcal{E}) = r$ for every $[f] \in H$ or $\deg(f^*\mathcal{E}) = r+1$ for every $[f] \in H$. That \mathcal{E} splits in one of the above two ways on every (i.e., not just general) $[f] \in H$ is forced by the fact that $f^*\mathcal{E}$ is ample and its rank and degree differ by at most 1. \square

Corollary 3.3. *Let X and \mathcal{E} be as above. Unless $r = 1$ and $f^*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(2)$, X admits an unsplit minimal dominating family of rational curves.*

Proof. Let H be a minimal dominating family of rational curves on X , $[f] \in H$ a general member. By Lemma 3.2,

$$r = \text{rank}(f^*\mathcal{E}) \leq \deg(f^*\mathcal{E}) = r \text{ or } r+1$$

When $r > 1$ or when $r = 1$ and $f^*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$, the above inequality shows that it is impossible for the curve parameterized by $[f]$ to split as a sum of two or more rational curves C_1, C_2, \dots, C_k : On the one hand $r = \text{rank}(f^*\mathcal{E}) \leq \deg(\mathcal{E}|_{C_i})$ for each C_i by ampleness of \mathcal{E} . On the other hand, the sum of the degrees of the $\mathcal{E}|_{C_i}$ must equal r or $r+1$. Therefore H is unsplit. \square

4. PROOF OF THEOREM 1.1

Let X be a smooth complex projective variety of dimension n , \mathcal{E} an ample vector bundle of rank r on X , and $p \leq r$ a positive integer such that $\wedge^p \mathcal{E} \subseteq \wedge^p T_X$. By Theorem 1.4, one may assume that $r > 1$. Let H be an unsplit minimal dominating family of rational curves on X guaranteed by Corollary 3.3. Lemma 3.2 shows that there are two possible ways for the vector bundle \mathcal{E} to split on the curves parameterized by H ; I address each case separately:

CASE I: First assume that $f^*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$ for every $[f] \in H$. The following result of Andreatta-Wisniewski deals with this situation:

Theorem 4.1. [AW01, 1.2] *Let X be a smooth complex projective variety such that $\rho(X) = 1$, \mathcal{E} a vector bundle of rank r on X , and H an unsplit minimal dominating family of rational curves on X . If there exists an integer a such that $f^*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r}$ for every $[f] \in H$, then*

there is a uniquely defined line bundle \mathcal{L} on X such that $\deg(f^*\mathcal{L}) = a$ and $\mathcal{E} \simeq \mathcal{L}^{\oplus r}$.

Remark 4.2. I was unable to follow all of the argument made in [AW01, 1.2], therefore an alternative proof is provided below. The method of lifting rational curves to $\mathbb{P}(\mathcal{E})$ remains the same as the proof given in [AW01]; modifications were made to reflect the fact that a general fiber of a rationally connected quotient may not have Picard number 1. (See 2.4-2.5 for more.) In fact, Theorem 4.5 is a generalization of the original statement. Since then, M. Andreatta has explained to me a nice fix for the apparent gap in the original proof of [AW01, 1.2].

Theorem 4.3. *Let X be a smooth complex projective variety of dimension n , \mathcal{E} a vector bundle of rank r on X , and H_1, H_2, \dots, H_k a collection of families of rational curves on X such that X is (H_1, H_2, \dots, H_k) -rationally connected. If there exists an integer $a \in \mathbb{Z}$ such that $f^*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r}$ for every $[f] \in H_1, H_2, \dots, H_k$, then there exists a finite surjective morphism $q : Y \rightarrow X$ from a variety Y such that:*

- (4.3.1) *There is a collection of families $V_1, V_2, \dots, V_l \subseteq \text{RatCurves}^n(Y)$ and a proper surjective morphism $q_* : \bigcup_{i=1}^l V_i \rightarrow \bigcup_{j=1}^k H_j$ where $q_*([\widehat{f}]) = [f]$ is given by $q \circ \widehat{f} = f$. The variety Y is (V_1, V_2, \dots, V_l) -rationally connected.*
- (4.3.2) *There is a (uniquely defined) line bundle \mathcal{L} on Y such that $\deg(f^*\mathcal{L}) = a$ and $q^*\mathcal{E} \simeq \mathcal{L}^{\oplus r}$.*

Proof. The argument applies induction with respect to r . Let $p : \mathbb{P}(\mathcal{E}) \rightarrow X$ be the projectivization of \mathcal{E} with relative tautological bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. For any $[f] \in H_1, H_2, \dots, H_k$ and $y \in p^{-1}(f(0))$ there is a unique lift $\widehat{f} : \mathbb{P}^1 \rightarrow \mathbb{P}(\mathcal{E})$ with the property that $p \circ \widehat{f} = f$ and $\deg(\widehat{f}^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = a$, $\widehat{f}(0) = y$: Since $\mathbb{P}(f^*\mathcal{E}) = \mathbb{P}^1 \times \mathbb{P}^{r-1}$, the morphism \widehat{f} is obtained by composing $\mathbb{P}(f^*\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E})$ with the morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \{y\} \subset \mathbb{P}^1 \times \mathbb{P}^{r-1}$. Thus, for a generic f , $\widehat{f}^*T_{\mathbb{P}(\mathcal{E})} = f^*T_X \oplus \mathcal{O}^{\oplus(r-1)}$.

For each $1 \leq i \leq k$, one may choose an irreducible component $\widehat{H}_i \subset \text{RatCurves}^n(\mathbb{P}(\mathcal{E}))$ parameterizing these lifts such that \widehat{H}_i dominates H_i . In fact, there exists a natural morphism $p_* : \text{RatCurves}^n(\mathbb{P}(\mathcal{E})) \rightarrow \text{RatCurves}^n(X)$ defined by $p_*(\widehat{f}) = p \circ \widehat{f}$.

Claim 4.3.3. *For each $1 \leq i \leq k$, the morphism $p_* : \widehat{H}_i \rightarrow H_i$ is proper and thus surjective.*

Proof. The proof uses the valuative criterion of properness [Har77, II.4.7]. Let B be the spectrum of a discrete valuation ring (or a germ of a smooth curve in the analytic context) with a closed point δ and a general point B^0 . Then for any family of morphisms $F_B : B \times \mathbb{P}^1 \rightarrow X$ coming from $B \rightarrow H_i$ one has $\mathbb{P}(F_B^* \mathcal{E}) = B \times \mathbb{P}^1 \times \mathbb{P}^{r-1}$. Now take $\widehat{F}_{B^0} : B^0 \times \mathbb{P}^1 \rightarrow \mathbb{P}(\mathcal{E})$, coming from a lift $B^0 \rightarrow \widehat{H}_i$ of $B \rightarrow H_i$. By construction \widehat{F}_{B^0} is the composition of $\mathbb{P}(F_B^* \mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E})$ with the product $id \times \psi_0 : B^0 \times \mathbb{P}^1 \rightarrow (B^0 \times \mathbb{P}^1) \times \mathbb{P}^{r-1}$, for some constant morphism $\psi_0 : B^0 \rightarrow \mathbb{P}^{r-1}$. The morphism ψ_0 extends trivially to $\psi : B \rightarrow \mathbb{P}^{r-1}$, thus \widehat{F}_{B^0} extends to \widehat{F}_B which is the composition of $\mathbb{P}(F_B^* \mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E})$ with the product $id \times \psi$, hence p_* is proper. \square

Continuing the proof of Theorem 4.3, consider the $(\widehat{H}_1, \widehat{H}_2, \dots, \widehat{H}_k)$ -rationally connected quotient of $\mathbb{P}(\mathcal{E})$, and let $Y \subset \mathbb{P}(\mathcal{E})$ be a general fiber. Notice that $\widehat{H}_1, \widehat{H}_2, \dots, \widehat{H}_k$ restricts to a collection of families $\widehat{H}_{Y_1}, \widehat{H}_{Y_2}, \dots, \widehat{H}_{Y_m} \subseteq \text{RatCurves}^n(Y)$, and Y is $(\widehat{H}_{Y_1}, \widehat{H}_{Y_2}, \dots, \widehat{H}_{Y_m})$ -rationally connected by construction. Also note that Y is projective and smooth.

Since X is (H_1, H_2, \dots, H_k) -rationally connected and $p_* : \widehat{H}_i \rightarrow H_i$ is surjective for each $1 \leq i \leq k$, the restriction map $p_Y : Y \rightarrow X$ is surjective.

Claim 4.3.4. *The morphism p_Y has no positive dimensional fiber, hence it is a finite morphism.*

Proof. By [Kol96, II.4.4], the morphism $p : \mathbb{P}(\mathcal{E}) \rightarrow X$ induces a surjective map

$$(4.3.5) \quad A_1(\mathbb{P}(\mathcal{E}))_{\mathbb{Q}} \xrightarrow{p_*} A_1(X)_{\mathbb{Q}} \rightarrow 0.$$

Let d be the dimension of $A_1(X)_{\mathbb{Q}}$. Then the dimension of $A_1(\mathbb{P}(\mathcal{E}))_{\mathbb{Q}}$ is $d + 1$ [Kol96, II.4.5], [Har77, Ex. II.7.9], and the kernel of p_* is the one dimensional space of 1-cycles in $A_1(\mathbb{P}(\mathcal{E}))_{\mathbb{Q}}$ that are contained in the fibers of p . Since the fibers of p are projective spaces, these 1-cycles are each rationally equivalent to a line in a fiber of p . Therefore, since X is rationally connected, they must be rationally equivalent in $A_1(\mathbb{P}(\mathcal{E}))_{\mathbb{Q}}$. If by contradiction there exists a proper curve $C \subset Y$ contracted by p_Y , one may take C as a generator for the kernel of p_* .

Now by [Kol96, IV.3.13.3], $A_1(X)_{\mathbb{Q}}$ is generated by the classes of curves parameterized by H_1, H_2, \dots, H_k , and $A_1(Y)_{\mathbb{Q}}$ is generated by the classes of curves parameterized by $\widehat{H}_{Y_1}, \widehat{H}_{Y_2}, \dots, \widehat{H}_{Y_m}$. Therefore one may choose lifts of d curves from H_1, H_2, \dots, H_k , say $\widehat{C}_1, \widehat{C}_2, \dots, \widehat{C}_d$, such that $\widehat{C}_i \subset Y$ for $1 \leq i \leq d$, and $A_1(\mathbb{P}(\mathcal{E}))_{\mathbb{Q}}$ is generated by

$\widehat{C}_1, \widehat{C}_2, \dots, \widehat{C}_d$ and C . But $C \subset Y$ by assumption, so C is a \mathbb{Q} -linear combination of $\widehat{C}_1, \widehat{C}_2, \dots, \widehat{C}_d$. This implies that $A_1(\mathbb{P}(\mathcal{E}))_{\mathbb{Q}}$ can be generated by d elements, a contradiction. Therefore p_Y does not contract any proper curve in Y , hence it is a finite morphism as desired. \square

Now consider the pullback $\widetilde{p} : \mathbb{P}(p_Y^* \mathcal{E}) \longrightarrow Y$ with the induced morphism $\widetilde{p}_Y : \mathbb{P}(p_Y^* \mathcal{E}) \longrightarrow \mathbb{P}(\mathcal{E})$ such that $p \circ \widetilde{p}_Y = p_Y \circ \widetilde{p}$. By the universal property of the fiber product the projective bundle \widetilde{p} admits a section $s : Y \longrightarrow \mathbb{P}(p_Y^* \mathcal{E})$ such that $\widetilde{p}_Y \circ s$ is the embedding of Y into $\mathbb{P}(\mathcal{E})$. This induces a sequence of bundles over Y :

$$(4.3.6) \quad 0 \longrightarrow \mathcal{E}' \longrightarrow p_Y^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_Y \longrightarrow 0$$

where \mathcal{E}' is a bundle of rank $r - 1$ on Y . In order to apply the inductive hypothesis to \mathcal{E}' , it suffices to show that \mathcal{E}' splits in the desired way on the curves parameterized by $\widehat{H}_{Y_1}, \widehat{H}_{Y_2}, \dots, \widehat{H}_{Y_m}$: First notice that $\deg(f^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_Y) = a$ for any curve $[f] \in \widehat{H}_{Y_1}, \widehat{H}_{Y_2}, \dots, \widehat{H}_{Y_m}$. (This follows from the fact that $\deg(\widehat{f}^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = a$ for every $[\widehat{f}] \in \widehat{H}_1, \widehat{H}_2, \dots, \widehat{H}_k$ as stated at the beginning of the proof.) Therefore, by restricting (4.3.6) to any $[\widehat{f}] \in \widehat{H}_{Y_1}, \widehat{H}_{Y_2}, \dots, \widehat{H}_{Y_m}$, one has:

$$0 \longrightarrow \widehat{f}^* \mathcal{E}' \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a) \longrightarrow 0$$

Twisting this sequence by $\mathcal{O}_{\mathbb{P}^1}(-a - 1)$ yields:

$$0 \longrightarrow \widehat{f}^* \mathcal{E}'(-a - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0$$

Now, one may write $\widehat{f}^* \mathcal{E}'(-a - 1) \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(\beta_i)$ where $\sum_{i=1}^{r-1} \beta_i = -(r - 1)$. The inclusion $\widehat{f}^* \mathcal{E}'(-a - 1) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r}$ implies that $\widehat{f}^* \mathcal{E}'(-a - 1)$ has no global sections, hence $\beta_i < 0$ for $1 \leq i \leq r - 1$. But since $\sum_{i=1}^{r-1} \beta_i = -(r - 1)$, $\beta_i = -1$ for all $1 \leq i \leq r - 1$. It follows that $\widehat{f}^* \mathcal{E}' \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r-1}$ for every $[\widehat{f}] \in \widehat{H}_{Y_1}, \widehat{H}_{Y_2}, \dots, \widehat{H}_{Y_m}$. Now let $q' : Y' \longrightarrow Y$ be the finite surjective morphism obtained from induction, and V_1, V_2, \dots, V_l the corresponding collection of families of rational curves in $\text{RatCurves}^n(Y')$ satisfying the conditions in 1. Pulling back the exact sequence (4.3.6) to Y' one obtains:

$$(4.3.7) \quad 0 \longrightarrow \mathcal{L}^{\oplus r-1} \longrightarrow (q' \circ p_Y)^* \mathcal{E} \longrightarrow \mathcal{L}' \longrightarrow 0$$

where \mathcal{L} is the uniquely defined line bundle coming from induction, and $\mathcal{L}' = q'^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{Y'}$ for simplicity. (Note that (4.3.7) is exact since the sheaves in (4.3.6) are each locally free.) I claim that $\mathcal{L} \simeq \mathcal{L}'$ as line bundles on Y' : First notice that \mathcal{L} and \mathcal{L}' agree on all of the rational curves parameterized by V_1, V_2, \dots, V_l . Indeed, for any $[f'] \in$

V_1, V_2, \dots, V_l :

$$f'^* \mathcal{L}^{\oplus r-1} = (f' \circ q')^* \mathcal{E} = \widehat{f}^* \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r-1}$$

and

$$f'^* \mathcal{L}' = (f' \circ q')^* \mathcal{O}_{\mathbb{P}^1}(1)|_Y = \widehat{f}^* \mathcal{O}_{\mathbb{P}^1}(1)|_Y \simeq \mathcal{O}_{\mathbb{P}^1}(a)$$

where $[\widehat{f}] \in \widehat{H}_{Y_1}, \widehat{H}_{Y_2}, \dots, \widehat{H}_{Y_m}$ is the image of $[f']$ under the map q'_* given in 1. Now, $N_1(Y')$ is generated by the classes of curves coming from V_1, V_2, \dots, V_l [Kol96, IV.3.13.3] and there exists a nondegenerate bilinear pairing

$$N_1(Y') \times N^1(Y') \longrightarrow \mathbb{Q}$$

given by the intersection number of curves and divisors. Since the pairing is nondegenerate, it follows that $\mathcal{L}^{-1} \otimes \mathcal{L}'$ is numerically equivalent to $\mathcal{O}_{Y'}$, and therefore $\mathcal{L}^{-1} \otimes \mathcal{L}'$ is torsion [Laz04, 1.1.20]. Let $\text{Spec}(\mathcal{A}) \rightarrow Y'$ be the unramified cyclic cover of Y' induced by the $\mathcal{O}_{Y'}$ -algebra $\mathcal{A} = \mathcal{O}_{Y'} \oplus (\mathcal{L}^{-1} \otimes \mathcal{L}') \oplus (\mathcal{L}^{-1} \otimes \mathcal{L}')^{\otimes 2} \oplus \dots \oplus (\mathcal{L}^{-1} \otimes \mathcal{L}')^{\otimes m-1}$, where m is the smallest positive integer such that $(\mathcal{L}^{-1} \otimes \mathcal{L}')^{\otimes m} = \mathcal{O}_{Y'}$. By the inductive assumption, Y' is rationally connected, hence simply connected, therefore $\text{Spec}(\mathcal{A}) \rightarrow Y'$ must be trivial. Therefore $m = 1$ and $(\mathcal{L}^{-1} \otimes \mathcal{L}') \simeq \mathcal{O}_{Y'}$ as desired.

Lastly, since Y' is rationally connected, [Kol96, IV.3.8] implies that $0 = H^0(Y', \Omega_{Y'}^1) \simeq H^1(Y', \mathcal{O}_{Y'})$, and therefore the sequence 4.3.7 splits. In other words, $(q' \circ p_Y)^* \mathcal{E} \simeq \mathcal{L}^{\oplus r}$ on Y' , and this completes the proof of Theorem 4.3. \square

Lemma 4.4. [AW01, 1.2.2] *Let X be a smooth complex projective Fano variety with $p : \mathbb{P}(\mathcal{E}) \rightarrow X$ a projectivization of a rank r bundle. Suppose that $\Psi : Y \rightarrow X$ is a finite morphism. If $\mathbb{P}(\Psi^*(\mathcal{E})) \simeq Y \times \mathbb{P}^{r-1}$ then $\mathbb{P}(\mathcal{E}) \simeq X \times \mathbb{P}^{r-1}$. \square*

Theorem 4.5. *Let X be a smooth complex projective Fano variety, \mathcal{E} a vector bundle of rank r on X , and $H_1, H_2, \dots, H_k \subseteq \text{RatCurves}^n(X)$ a collection of rational curves such that X is (H_1, H_2, \dots, H_k) -rationally connected. If there exists an integer $a \in \mathbb{Z}$ such that $f^* \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r}$ for every $[f] \in H_1, H_2, \dots, H_k$, then there is a uniquely defined line bundle \mathcal{L} on X such that $\deg(f^* \mathcal{L}) = a$ and $\mathcal{E} \simeq \mathcal{L}^{\oplus r}$.*

Proof. This is immediate from Theorem 4.3 and Lemma 4.4. \square

Remark 4.6. When X is both uniruled and $\rho(X) = 1$, X must be Fano. Therefore Theorem 4.1 follows from Theorem 4.5.

Continuing with the proof of Theorem 1.1, use Theorem 4.5 to define a new vector bundle $\mathcal{F} := \mathcal{L}^{\oplus p}$ on X , and note that in our case \mathcal{L} is

ample. Recall that $p \leq r$ by assumption, therefore $\mathcal{F} \subseteq \mathcal{E}$. It follows that $\mathcal{L}^{\otimes p} = \det(\mathcal{F}) \subseteq \wedge^p \mathcal{E} \subseteq \wedge^p T_X$. By [ADK08, 6.3] (= Theorem 1.7), either $X \simeq \mathbb{P}^n$ or $X \simeq Q_p \subseteq \mathbb{P}^{p+1}$.

CASE II: Now assume that $\rho(X) = 1$ and consider the case that $f^* \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}$ for every $[f] \in H$. I will show that there exists a vector bundle injection $f^* \mathcal{E} \hookrightarrow f^* T_X^+$, and use this fact to study the geometry of the variety of minimal rational tangents \mathcal{C}_x at general points $x \in X$.

Lemma 4.7. *Let X be a smooth complex projective variety of dimension n , and let \mathcal{E} be an ample vector bundle of rank r on X such that $\wedge^p \mathcal{E} \subseteq \wedge^p T_X$ for some positive integer $p \leq r$. Let H be a minimal dominating family of rational curves on X , and let $[f] \in H$ be a general member. If $f^* \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}$, then there exists a vector bundle injection $f^* \mathcal{E} \hookrightarrow f^* T_X^+$.*

Proof. Since $[f] \in H$ is a general member, the splitting type of T_X on the curve parameterized by $[f]$ is $f^* T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus (n-d-1)}$ [Kol96, IV.2.9, IV.2.10]. When $f^* \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}$, a simple counting argument shows that $r - 1 \leq d$: If $f^* \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}$ then $f^*(\wedge^p \mathcal{E})$ splits as a sum of line bundles of which exactly $\binom{r-1}{p-1}$ have degree $p + 1$. A similar computation shows that the direct sum decomposition of $f^*(\wedge^p T_X)$ includes exactly $\binom{d}{p-1}$ line bundles of degree $p + 1$. Since $p + 1$ is the largest degree of any line bundle occurring in the decomposition of $f^*(\wedge^p T_X)$ and since I assume $f^*(\wedge^p \mathcal{E}) \subseteq f^*(\wedge^p T_X)$, it follows that $r - 1 \leq d$. But then $f^* \mathcal{E} \hookrightarrow f^* T_X^+ \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d}$ as desired. \square

Now let $x \in X$ be a general point, H_x the normalization of the subscheme of H parameterizing curves passing through $x \in X$, and $\tau_x : H_x \rightarrow \mathcal{C}_x \subseteq \mathbb{P}(T_x X)$ the tangent map defined in Section 2. Let H_x^i , $1 \leq i \leq k$, be the irreducible components of H_x , and define $\mathcal{C}_x^i := \text{im}(\tau_x(H_x^i))$. Fix an irreducible component H_x^i and let $[f] \in H_x^i$ be a general member $f : \mathbb{P}^1 \rightarrow X$ such that $f(o) = x$ for a point $o \in \mathbb{P}^1$. The fiber $(f^* T_X)_o$ of $f^* T_X$ over the point o is naturally isomorphic to $T_x X$, and under this isomorphism the positive part $(f^* T_X)_o^+ \subset (f^* T_X)_o$ cuts out a $(d+1)$ -dimensional linear subspace $T_x X_f^+ \subseteq T_x X$. By Lemma 4.7, $(f^* \mathcal{E})_o \hookrightarrow (f^* T_X)_o^+$, and this induces the inclusion $\mathcal{E}_x \subseteq T_x X_f^+$. By [Hwa01, 2.3] (= Lemma 2.8), it follows that $\mathbb{P}(\mathcal{E}_x) \subseteq \overline{T_{\tau_x([f])} \mathcal{C}_x^i} \subseteq \mathbb{P}(T_x X)$. Now the argument in [Ara06, 4.1, 4.2, 4.3] implies that \mathcal{C}_x^i is a linear subspace of $\mathbb{P}(T_x X)$; I include an outline of the main steps here for the convenience of the reader: By Lemma 4.8

below, the inclusion $\mathbb{P}(\mathcal{E}_x) \subseteq \overline{T_{\tau_x([f])}\mathcal{C}_x^i}$ forces \mathcal{C}_x^i to have the structure of a cone in $\mathbb{P}(T_x X)$ with $\mathbb{P}(\mathcal{E}_x)$ contained in its vertex. Now the result follows from Lemma 4.9 and the fact that H_x is smooth [Kol96, II.1.7, II.2.16] and $\tau_x : H_x \rightarrow \mathcal{C}_x$ is the normalization morphism ([Keb02], [HM04] = Theorem 2.6.)

Lemma 4.8. [Ara06, 4.2] *Let Z be an irreducible closed subvariety of \mathbb{P}^m . Assume that there is a dense open subset U of the smooth locus of Z and a point $z_0 \in \mathbb{P}^m$ such that $z_0 \in \bigcap_{z \in U} T_z Z$. Then Z is a cone in \mathbb{P}^m and z_0 lies in the vertex of this cone.*

Lemma 4.9. [Ara06, 4.3] *If Z is an irreducible cone in \mathbb{P}^m and the normalization of Z is smooth, then Z is a linear subspace of \mathbb{P}^m .*

From here one can conclude that the irreducible components of \mathcal{C}_x are all linear subspaces of $\mathbb{P}(T_x X)$. The following proposition of J.M. Hwang shows that in this case \mathcal{C}_x is actually irreducible, thus itself a linear subspace of $\mathbb{P}(T_x X)$:

Proposition 4.10. [Hwa01, 2.2] *Let X be a smooth complex projective variety, H a minimal dominating family of rational curves on X , and $\mathcal{C}_x \in \mathbb{P}(T_x X)$ the corresponding variety of minimal rational tangents at $x \in X$. Assume that for a general $x \in X$, \mathcal{C}_x is a union of linear subspaces of $\mathbb{P}(T_x X)$. Then the intersection of any two irreducible components of \mathcal{C}_x is empty.*

Now since H_x is the normalization of \mathcal{C}_x and it is dimension $d := \deg(f^*T_X) - 2$ for a general point $x \in X$, \mathcal{C}_x is in fact a linear subspace of $\mathbb{P}(T_x X)$ of dimension d for every general point $x \in X$. Therefore one can apply the main theorem of [Ara06] (= Theorem 2.7) to conclude that the H -rationally connected quotient $\pi^\circ : X^\circ \rightarrow Y^\circ$ admits the structure of a projective space bundle. But since the Picard number of X is 1, Y° is a point by Proposition 2.1. Therefore $X \simeq \mathbb{P}^n$ as desired, and this proves Theorem 1.1. \square

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